

IMPROVED ANALYSIS OF ALGORITHMS BASED ON SUPPORTING HALFSPACES AND QUADRATIC PROGRAMMING FOR THE CONVEX INTERSECTION AND FEASIBILITY PROBLEMS

C.H. JEFFREY PANG

ABSTRACT. This paper improves the algorithms based on supporting halfspaces and quadratic programming for convex set intersection problems in our earlier paper in several directions. First, we give conditions so that much smaller quadratic programs (QPs) and approximate projections arising from partially solving the QPs are sufficient for multiple-term superlinear convergence for nonsmooth problems. Second, we identify additional regularity, which we call the second order supporting hyperplane property (SOSH), that gives multiple-term quadratic convergence. Third, we show that these fast convergence results carry over for the convex inequality problem. Fourth, we show that infeasibility can be detected in finitely many operations. Lastly, we explain how we can use the dual active set QP algorithm of Goldfarb and Idrani to get useful iterates by solving the QPs partially, overcoming the problem of solving large QPs in our algorithms.

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1. INTRODUCTION

We consider two different problems in this paper. First, let K_1, \dots, K_r be r closed convex sets in a Hilbert space X . The *Set Intersection Problem* (SIP) is

$$(SIP): \quad \text{Find } x \in K := \bigcap_{i=1}^r K_i, \text{ where } K \neq \emptyset. \quad (1.1)$$

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The *Convex Inequality Problem* (CIP) is

$$(CIP): \quad \text{For a convex } f : \mathbb{R}^n \rightarrow \mathbb{R}, \text{ find } x \in \mathbb{R}^n \text{ s.t. } f(x) \leq 0. \quad (1.2)$$

This paper improves on the results in [Pan14b], where we studied convergence results for accelerating convergence of algorithms for the SIP. The idea there was to collect as many supporting halfspaces generated by the projection process to create a polyhedron that is an outer approximation of K . Then one can project onto this polyhedron using quadratic programming. See Figure 1.1.

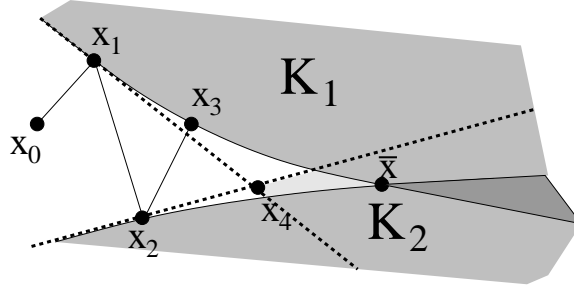


FIGURE 1.1. The method of alternating projections on two convex sets K_1 and K_2 in \mathbb{R}^2 with starting iterate x_0 arrives at x_3 in three iterations. Consider the supporting halfspaces planes of K_1 and K_2 at x_1 and x_2 . The projection of x_1 onto the intersection of these halfspaces, which is x_4 , is much closer to the point \bar{x} than x_3 , especially when the boundary of K_1 and K_2 have fewer second order effects and when the angle between the boundary of K_1 and K_2 is small. On the other hand, the point x_3 is ruled out by the supporting hyperplane of K_2 passing through x_2 .

We note that the idea of supporting halfspaces and quadratic programming was studied in [GP98, GP01], but for the CIP when $f(\cdot)$ is the maximum of a finite number of smooth functions. Quadratic programs with one affine constraint (not necessarily of codimension 1) and a halfspace were used to accelerate algorithms for the CIP in [Pie84, BCK06]. The idea of using QPs was also present in other works on the CIP (for example [Fuk82]).

A popular method for solving the SIP is the method of alternating projections. We highlight the references [BB96, BR09, Cen84, CZ97, Com93, Com96, Deu95, Deu01a, ER11], as well as [Deu01b, Chapter 9] and [BZ05, Subsubsection 4.5.4], for an introduction on the SIP and their applications. The papers [GPR67, GK89, BDHP03] explored acceleration methods for the method of alternating projections. Another acceleration method is the Dos Santos method [San87, Pie81], which is based on Cimmino's method for linear equations [Cim38].

We remark that the treatment for the case when all the sets are affine spaces are covered in [Pan14a].

In this paper, we deal with the case where the $f(\cdot)$ in the CIP were convex but not smooth. Some early work on (not necessarily convex) inequality problems are [Rob76, PM79, MPH81, Fuk82], and we elaborate on their contributions in this introduction. A related work is [FL03], where SQP methods are used to solve nonconvex but smooth CIPs.

In [Rob76], Robinson considered the *K-Convex Inequality Problem* (KCIP), which is a generalization of the (CIP). For $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, and a closed convex cone $K \subset \mathbb{R}^m$, we write $y_1 \leq_K y_2$ if $y_2 - y_1 \in K$. The KCIP is defined by

$$\text{(KCIP): For } f : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ and } C \subset \mathbb{R}^n, \text{ find } x \in C \text{ s.t. } f(x) \leq_K 0. \quad (1.3)$$

Robinson's algorithm in [Rob76] for the CIP can be described as follows: At each iterate x_i , a subgradient $y_i \in \partial f(x_i)$ is obtained, and the halfspace

$$H_i^\leq := \{x \in \mathbb{R}^n \mid f(x_i) + \langle y_i, x - x_i \rangle \leq 0\} \quad (1.4)$$

contains $f^{-1}((-\infty, 0])$. The next iterate x_{i+1} is obtained by projecting x_i onto H_i^\leq . Assuming regularity and convexity (and no smoothness), Robinson proved that the CIP (or more generally, the KCIP) converges at least linearly.

The idea of collecting halfspaces and projecting onto their intersection using quadratic programming can be carried over for the CIP. In the CIP, the halfspaces are of type (1.4).

A related paper on the CIP is [Fuk82], where Fukushima obtained finite convergence for the CIP when $f^{-1}((-\infty, 0])$ has nonempty interior, assuming only convexity and not smoothness. The idea is to try to find an x satisfying

$$f(x) \leq -\epsilon_k$$

at iteration k , where $\{\epsilon_k\}$ is a sequence of positive numbers converging to zero at a rate slower than any linearly converging sequence. (It appears that [PI88] have come up with a similar result independently.) For smooth problems, this idea can be traced back to [PM79, MPH81] or possibly earlier.

Other papers on the SIP are [Kiw95], where the interest is on problems where r , the number of closed convex sets K_i , is large. A method for the best approximation problem (stated in (5.1)) is Dykstra's algorithm [Dyk83, BD85, Han88]. There has been recent interest in nonconvex SIP problems [LM08, LLM09]. We believe that an adaptation of our algorithm can be useful for nonconvex problems.

It appears that prevailing algorithms for the SIP (see for example the algorithms in [CCC⁺12, ER11]) do not exploit smoothness of the sets and fall back to a Newton-like method and achieve superlinear convergence in the manner of [GP98, GP01] and the algorithms of this paper. We note however that variants of the algebraic reconstruction technique (ART), which try to find a point in the intersection of hyperslabs rather than general convex sets, can achieve finite convergence (See [HC08] and the references therein).

1.1. Contributions of this paper. This paper improves on the algorithm for the SIP in [Pan14b]. In [Pan14b], a multiple-term superlinearly convergent algorithm for non-smooth SIPs for the case when $X = \mathbb{R}^n$ was proposed, but the algorithm there requires one to solve impractically huge QPs. In this paper, we show that the following adjustments, reflected in Algorithm 3.1, maintain such fast convergence:

- Instead of accumulating $r\bar{p}$ halfspaces (where \bar{p} is a huge parameter) as proposed in [Pan14b], superlinear convergence can be achieved if the normals of two of the halfspaces produced by the projection process are close enough to each other. (See Theorem 3.10.) This condition is weaker and easier to check in practice, and can greatly reduce the size of the QPs that we need to solve to maintain superlinear convergence. (See Remark 3.2.) We present Corollaries 3.11 and 3.12 based on this result. In particular, Corollary 3.12 states

that when the boundaries of the sets are smooth, our algorithm reduces to a Newton method, and our framework can prove that the convergence is indeed superlinear or quadratic.

- The requirement of projecting onto the intersection of the halfspaces is relaxed, reflecting that an approximate projection can also guarantee superlinear convergence. The quadratic programming subproblem for projecting onto the polyhedron can be solved partially using a dual active set algorithm of [Gl83], then extrapolated to a feasible point in the polyhedron. See Section 6 and Remark 3.3.

Large parts of the proofs in [Pan14b] for this result remain the same here. We only focus on the additional details without repeating the proofs that are largely unchanged from [Pan14b].

We introduce the *Second Order Supporting Hyperplane* (SOSH) property (Definition 3.5), which we show is present in sets defined by convex inequalities (Proposition 3.6). Moreover, the SOSH property is preserved under intersections under a constraint qualification (Proposition 3.7). Under the SOSH property, we can achieve multiple-term quadratic convergence of the SIP algorithm.

We then show that a multiple-term superlinear convergence for a nonsmooth CIP (1.2) can be achieved using the techniques studied for the SIP in Section 4.

Next, we look at infeasibility detection. In [Pan14b, Section 6], we had discussed infeasibility detection for the Best Approximation Problem (BAP) (See (5.1)). In Theorem 5.2, we show that under reasonable conditions, algorithms for the SIP, CIP and BAP do not have strong cluster points in the infeasible case in finite dimensions, and can even obtain a certificate of infeasibility in a finite number of operations. We make further observations about the BAP in Theorem 5.7.

Lastly, we explain that the dual active set QP algorithm of Goldfarb and Idrani [Gl83] gives good iterates after each inner iteration even when the QPs in our algorithms are not solved fully. Such a property is useful since the QPs that we solve may be large and difficult to solve to optimality.

1.2. Notation. We recall some standard notation in convex analysis that are helpful for the rest of the paper. As our results rely on the compactness of the unit sphere, we only treat the finite dimensional case here. Let $C \subset \mathbb{R}^n$ be a closed convex set, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function, and $x \in C$. Then we have the following notation:

$N_C(x)$	The <i>normal cone</i> $N_C(x)$ at the point $x \in C$ is the set $\{v \mid \langle v, y - x \rangle \leq 0 \text{ for all } y \in C\}$.
$\mathbb{B}(x, r)$	Ball with center x and radius r : $\mathbb{B}(x, r) := \{y \mid \ y - x\ \leq r\}$. We write $\mathbb{B} := \mathbb{B}(0, 1)$.
$\partial f(x)$	The <i>subdifferential</i> of f at x : $\partial f(x) := \{y \mid f(x') \geq f(x) + \langle y, x' - x \rangle \text{ for all } x' \in \mathbb{R}^n\}$.
$d(x, S)$	The distance of x to a set $S \subset \mathbb{R}^n$: $d(x, S) := \inf_{s \in S} \ x - s\ $.
$\text{lip } f(x)$	The <i>Lipschitz modulus</i> of f at x : $\text{lip } f(x) := \limsup_{\substack{x_1, x_2 \rightarrow x \\ x_1 \neq x_2}} \frac{\ f(x_1) - f(x_2)\ }{\ x_1 - x_2\ }$.
$\text{pos}(S)$	For a set $S \subset \mathbb{R}^n$, the <i>positive hull</i> is the set $\text{pos}(S) := \{ts \mid t \in [0, \infty), s \in S\}$.
$R(S)$	For a convex set $S \subset \mathbb{R}^n$, the <i>recession cone</i> is the set $\{d \mid x + td \in S \text{ for all } t \geq 0 \text{ and } x \in S\}$.

We denote $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ to be a *set-valued map* that maps a point in \mathbb{R}^n to a subset of \mathbb{R}^m . A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous* if its *graph*

$\text{Graph}(F) := \{(x, y) \mid y \in F(x)\}$ is closed. A convex cone $C \subset \mathbb{R}^n$ is *pointed* if it does not contain a line. The notation “ ∂ ” can also mean the boundary of a closed set, which should not lead to confusion with the subdifferential. In our proofs, we also make use of the Pompeiu Hausdorff distance. We refer the reader to standard texts in convex and variational analysis [Roc70, HUL93, RW98, Cla83, Mor06] for more information.

2. PRELIMINARY RESULTS

In this section, we collect a few results that are nonstandard, but will be useful for the rest of the paper.

Definition 2.1. (*Fejér monotone sequence*) Let X be a Hilbert space, $C \subset X$ be a closed convex set, and $\{x_i\}$ be a sequence in X . We say that $\{x_i\}$ is *Fejér monotone with respect to C* if

$$\|x_{i+1} - c\| \leq \|x_i - c\| \text{ for all } c \in C \text{ and } i = 1, 2, \dots$$

A tool for obtaining a Fejér monotone sequence is stated below.

Theorem 2.2. (*Fejér attraction property*) Let X be a Hilbert space. For a closed convex set $C \subset X$, $x \in X$, $\lambda \in [0, 2]$, and the projection $P_C(x)$ of x onto C , let the relaxation operator $R_{C,\lambda} : X \rightarrow X$ [Agm83] be defined by

$$R_{C,\lambda}(x) = x + \lambda(P_C(x) - x).$$

Then

$$\|R_{C,\lambda}(x) - c\|^2 \leq \|x - c\|^2 - \lambda(2 - \lambda)d(x, C)^2 \text{ for all } y \in C. \quad (2.1)$$

Here are some consequences of Fejér monotonicity. We take our results from [BZ05, Theorem 4.5.10 and Lemma 4.5.8].

Theorem 2.3. (*Properties of Fejér monotonicity*) Let X be a Hilbert space, let $C \subset X$ be a closed convex set and let $\{x_i\}$ be a Fejér monotone sequence with respect to C . Then

- (1) $\{x_i\}$ is bounded and $d(C, x_{i+1}) \leq d(C, x_i)$, and
- (2) $\{x_i\}$ has at most one weak cluster point in C .

The following result is elementary and proved in [Pan14b], and will also be used in the proof of Theorem 4.4. Recall that the dual cone $K^+ \subset \mathbb{R}^n$ of a convex cone $K \subset \mathbb{R}^n$ is

$$K^+ := \{y \mid \langle x, y \rangle \geq 0 \text{ for all } x \in K\}.$$

Lemma 2.4. (*Pointed cone*) For a closed pointed convex cone $K \subset \mathbb{R}^n$, there is a unit vector d in K^+ , the dual cone of K , and some $c > 0$ such that $\mathbb{B}(d, c) \subset K^+$. For any unit vector $v \in K$, we have $d^T v \geq c$.

Moreover, suppose $\lambda_i \geq 0$, and v_i are unit vectors in K for all i , and $\sum_{i=1}^{\infty} \lambda_i v_i$ converges to \bar{v} . Clearly, $\bar{v} \in K$. Then $\|\sum_{i=1}^{\infty} \lambda_i v_i\| \geq c \sum_{i=1}^{\infty} \lambda_i$, which also implies that $\sum_{i=1}^{\infty} \lambda_i$ is finite.

We now recall the definition of semismoothness.

Definition 2.5. [Mif77] (*Semismoothness*) Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex. We say that Φ is *semismooth at x* if it is directionally differentiable at x and for any $V \in \partial\Phi(x + h)$,

$$\Phi(x + h) - \Phi(x) - Vh = o(\|h\|).$$

We say that Φ is *strongly semismooth at x* if Φ is semismooth at x and

$$\Phi(x + h) - \Phi(x) - Vh = O(\|h\|^2).$$

Semismoothness is also defined for vector-valued functions that need not be convex, but this definition above is enough for our purposes. Moreover, it is proved that convexity implies semismoothness, so semismoothness is superfluous in our context. But we shall need to use strong semismoothness later for Theorem 4.4(b).

3. IMPROVING CONVERGENCE RESULTS OF THE SIP

In this section, we show how to improve the convergence results for the SIP in [Pan14b] as detailed in Subsection 1.1. We recall an adaptation of the \bar{p} -term super-linear convergent algorithm of [Pan14b] for the SIP.

Algorithm 3.1. (*SHQP algorithm for the SIP*) For a starting iterate $x_0 \in \mathbb{R}^n$ and closed convex sets $K_l \subset \mathbb{R}^n$, where $1 \leq l \leq r$, find a point in $K := \cap_{l=1}^r K_l$.

Step 0: Set $i = 0$, and let \bar{p} be a positive integer.

Step 1: For $l \in \{1, \dots, r\}$, define $x_i^{(l)} \in \mathbb{R}^n$, $a_i^{(l)} \in \mathbb{R}^n$ and $b_i^{(l)} \in \mathbb{R}$ by

$$\begin{aligned} x_i^{(l)} &= P_{K_l}(x_i), \\ a_i^{(l)} &= x_i - x_i^{(l)}, \\ \text{and } b_i^{(l)} &= \langle a_i^{(l)}, x_i^{(l)} \rangle. \end{aligned}$$

For each i , we let $l_i^* \in \{1, \dots, r\}$ be such that

$$l_i^* := \arg \max_{1 \leq l \leq r} \|x_i - P_{K_l}(x_i)\|.$$

Step 2: Choose $S_i \subset \{\max(i - \bar{p}, 0), \dots, i\} \times \{1, \dots, r\}$, and define $\tilde{F}_i \subset \mathbb{R}^n$ by

$$\tilde{F}_i := \bigcap_{(j,l) \in S_i} H_{(j,l)}^{\leq}, \quad (3.1)$$

$$\text{where } H_{(j,l)}^{\leq} := \{x : \langle a_j^{(l)}, x \rangle \leq b_j^{(l)}\}. \quad (3.2)$$

In other words, $H_{(j,l)}^{\leq}$ is the halfspace generated by projecting x_j onto K_l . Let x_{i+1} be chosen such that

- (1) $x_{i+1} \in \tilde{F}_i$,
- (2) (Fejér attraction) $\|x_{i+1} - c\| \leq \|x_i - c\|$ for all $c \in K$, and
- (3) $x_i - x_{i+1}$ lies in $\text{conv}(\{a_j^{(l)} : (j,l) \in S_i\})$.

Step 3: Set $i \leftarrow i + 1$, and go back to step 1.

We try to keep our notation consistent with that of [Pan14b], but we decided that it is better to use the index i in a different manner from [Pan14b].

Remark 3.2. (Choice of S_i) We leave the choice of S_i open in Algorithm 3.1. The choice $S_i = \{\max(i - \bar{p}, 0), \dots, i\} \times \{1, \dots, r\}$ was studied in [Pan14b]. We will also look at the choice $S_i = \{i\} \times \{1, \dots, r\}$ in Corollary 3.12.

Remark 3.3. (Step 2 of Algorithm 3.1) One way to obtain x_{i+1} is by projecting x_i onto \tilde{F}_i . Such an x_{i+1} would satisfy the conditions (1), (2) and (3) of step 2 by the properties of the projection. The argument to see how (2) is satisfied is simple: The polyhedron \tilde{F}_i contains K , and by the Fejér attractive property of projections, $\|x_{i+1} - c\| \leq \|x_i - c\|$ for all $c \in \tilde{F}_i$. Condition (3) follows from the KKT conditions of the projection operation.

A point satisfying the conditions (1), (2) and (3) may be easier to obtain than the projection. For example, one can use the dual active set quadratic programming

algorithm of Goldfarb and Idrani [Gl83] to obtain a point \tilde{x}_{i+1} that is the projection of x_i onto the polyhedron formed by intersecting a subset of $\{H_{(j,l)}^{\leq}\}_{(j,l) \in S_i}$. If the point $z := \lambda[\tilde{x}_{i+1} - x_i] + x_i$ for some $\lambda \in [1, 2]$ is such that $z \in \tilde{F}_i$, then in view of Theorem 2.2, x_{i+1} can be taken to be z . For more details on applying the dual quadratic programming algorithm to solve the SIP, we refer to Section 6.

Before we remark on the \bar{p} -term quadratic convergence of the algorithm in [Pan14b], we need to look at a theorem on convex sets proved in [Pan14b] and the SOSH property defined and studied afterward.

Theorem 3.4. [Pan14b] (*Supporting hyperplane near a point*) Suppose $C \subset \mathbb{R}^n$ is a closed convex set, and let $\bar{x} \in C$. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that for any point $x \in [\mathbb{B}(\bar{x}, \delta) \cap C] \setminus \{\bar{x}\}$ and supporting hyperplane A of C with unit normal $v \in N_C(x)$ at the point x , we have $\frac{d(\bar{x}, A)}{\|x - \bar{x}\|} \leq \epsilon$.

Since $d(\bar{x}, A) = -\langle v, \bar{x} - x \rangle$, the conclusion can be replaced by

$$0 \leq -\langle v, \bar{x} - x \rangle \leq \epsilon \|\bar{x} - x\|. \quad (3.3)$$

Definition 3.5. (Second order supporting hyperplane property) Suppose $C \subset \mathbb{R}^n$ is a closed convex set, and let $\bar{x} \in C$. We say that C has the *second order supporting hyperplane (SOSH) property at \bar{x}* (or more simply, C is SOSH at \bar{x}) if there are $\delta > 0$ and $M > 0$ such that for any point $x \in [\mathbb{B}_\delta(\bar{x}) \cap C] \setminus \{\bar{x}\}$ and $v \in N_C(x)$ such that $\|v\| = 1$, we have

$$0 \leq -\langle v, \bar{x} - x \rangle \leq M \|\bar{x} - x\|^2. \quad (3.4)$$

It is clear how (3.3) compares with (3.4). The next two results show that SOSH is prevalent in applications.

Proposition 3.6. (*Smoothness implies SOSH*) Suppose a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is C^2 at \bar{x} . Then the set $C = \{x \mid f(x) \leq 0\}$ is SOSH at \bar{x} .

Proof. Consider $\bar{x}, x \in C$. In order for the problem to be meaningful, we shall only consider the case where $f(\bar{x}) = 0$. We also assume that $f(x) = 0$ so that C has a supporting hyperplane at x . An easy calculation gives $N_C(\bar{x}) = \mathbb{R}_+ \{\nabla f(\bar{x})\}$ and $N_C(x) = \mathbb{R}_+ \{\nabla f(x)\}$. Convexity ensures that $0 \leq \frac{-\nabla f(x)(\bar{x} - x)}{\|\bar{x} - x\|^2}$ by Theorem 3.4.

Without loss of generality, let $\bar{x} = 0$. We have

$$f(x) = f(0) + \nabla f(0)x + \frac{1}{2}x^T \nabla^2 f(0)x + o(\|x\|^2)$$

Since $f(x) = f(0) = 0$ and $[\nabla f(0) - \nabla f(x)]x = x^T \nabla^2 f(0)x + o(\|x\|^2)$, we have

$$-\nabla f(x)(x) = [\nabla f(0) - \nabla f(x)]x + \frac{1}{2}x^T \nabla^2 f(0)x + o(\|x\|^2) = O(\|x\|^2).$$

Therefore, we are done. \square

Proposition 3.7. (*SOSH under intersection*) Suppose $K_l \subset \mathbb{R}^n$ are closed convex sets that are SOSH at \bar{x} for $l \in \{1, \dots, r\}$. Let $K := \cap_{l=1}^r K_l$, and suppose that

$$\sum_{l=1}^r v_l = 0, v_l \in N_{K_l}(\bar{x}) \text{ implies } v_l = 0 \text{ for all } l \in \{1, \dots, r\}. \quad (3.5)$$

Then K is SOSH at \bar{x} .

Proof. Since each K_l is SOSH at \bar{x} , we can find $\delta > 0$ and $M > 0$ such that for all $l \in \{1, \dots, r\}$ and $x \in K_l \cap \mathbb{B}_\delta(\bar{x})$ and $v \in N_{K_l}(x)$, we have

$$0 \leq -\langle v, \bar{x} - x \rangle \leq M \|v\| \|\bar{x} - x\|^2.$$

Claim 1: We can reduce $\delta > 0$ if necessary so that

$$\sum_{l=1}^r v_l = 0, v_l \in N_{K_l}(x) \quad (3.6)$$

implies $v_l = 0$ for all $l \in \{1, \dots, r\}$ and $x \in K \cap \mathbb{B}_\delta(\bar{x})$.

Suppose otherwise. Then we can find $\{x_i\}_{i=1}^\infty \in K$ such that $\lim x_i = \bar{x}$ and for all $i > 0$, there exists $v_{l,i} \in N_{K_l}(x_i)$ such that $\sum_{l=1}^r v_{l,i} = 0$ but not all $v_{l,i} = 0$. We can normalize so that $\|v_{l,i}\| \leq 1$, and for each i , $\max_l \|v_{l,i}\| = 1$. By taking a subsequence if necessary, we can assume that $\lim v_{l,i}$, say \bar{v}_l , exists for all l . Not all \bar{v}_l can be zero, but $\sum_{l=1}^r \bar{v}_l = 0$. The outer semicontinuity of the normal cone mapping implies that $\bar{v}_l \in N_{K_l}(\bar{x})$. This is a contradiction to (3.5), which ends the proof of Claim 1.

Claim 2: There exists a constant M' such that whenever $x \in \mathbb{B}_\delta(\bar{x}) \cap K$, $v_l \in N_{K_l}(x)$ and $v = \sum_{l=1}^r v_l$, then $\max \|v_l\| \leq M' \|v\|$.

Suppose otherwise. Then for each i , there exists $x_i \in \mathbb{B}_\delta(\bar{x}) \cap K$ and $\tilde{v}_{l,i} \in N_{K_l}(x_i)$ such that $\tilde{v}_i = \sum_{l=1}^r \tilde{v}_{l,i}$, $\|\tilde{v}_i\| \leq \frac{1}{i}$, and $\max_l \|\tilde{v}_{l,i}\| = 1$ for all i . As we take limits to infinity, this would imply that (3.6) is violated, a contradiction. This ends the proof of Claim 2.

Since (3.6) is satisfied, this means that $N_K(x) = \sum_{l=1}^r N_{K_l}(x)$ for all $x \in \mathbb{B}_\delta(\bar{x}) \cap K$ by the intersection rule for normal cones in [RW98, Theorem 6.42]. Then each $v \in N_K(x)$ can be written as a sum of elements in $N_{K_l}(x)$, say $v = \sum_{l=1}^r v_l$, where $v_l \in N_{K_l}(x)$, and $\max \|v_l\| \leq M' \|v\|$. Then

$$\begin{aligned} -\langle v, \bar{x} - x \rangle &= \sum_{l=1}^r -\langle v_l, \bar{x} - x \rangle \\ &\leq M \|\bar{x} - x\|^2 \sum_{l=1}^r \|v_l\| \leq M \|\bar{x} - x\|^2 r M' \|v\|. \end{aligned}$$

Thus we are done. \square

We recall the definition of local metric inequality, sometimes referred to as linear regularity.

Definition 3.8. (Local metric inequality) We say that a collection of closed sets K_l , $l = 1, \dots, r$ satisfies the *local metric inequality* at \bar{x} if there are $\bar{\kappa} > 0$ and $\delta > 0$ such that

$$d(x, \cap_{l=1}^r K_l) \leq \bar{\kappa} \max_{1 \leq l \leq r} d(x, K_l) \text{ for all } x \in \mathbb{B}_\delta(\bar{x}). \quad (3.7)$$

The following result is well-known, and we haven't been able to pin an original source. We refer to [Pan14b] for a discussion on its proof.

Lemma 3.9. (Condition for local metric inequality) Suppose $\bar{x} \in K$, $K_l \subset \mathbb{R}^n$ are closed convex sets for $l = 1, \dots, r$ and $K := \cap_{l=1}^r K_l$. Suppose that

(1) If $\sum_{l=1}^r v_l = 0$ for some $v_l \in N_{K_l}(\bar{x})$, then $v_l = 0$ for all $l = 1, \dots, r$.

Then $\{K_l\}_{l=1}^r$ satisfies the local metric inequality at \bar{x} for some $\bar{\kappa} \geq 0$.

We now prove our convergence result for Algorithm 3.1.

Theorem 3.10. (Convergence rates for the SIP) Suppose Algorithm 3.1 with parameter \bar{p} produces a sequence $\{x_i\}$ that converges to a point $\bar{x} \in K$, and the convergence is not finite. Suppose also that

(1) If $\sum_{l=1}^r v_l = 0$ for some $v_l \in N_{K_l}(\bar{x})$, then $v_l = 0$ for all $l = 1, \dots, r$.

In view of condition (1) and Lemma 3.9, $\{K_l\}_{l=1}^r$ satisfies the local metric inequality at \bar{x} with some constant, say $\bar{\kappa}$. Let $\bar{\alpha}$ be such that

$$\bar{\alpha} < \sin^{-1}(1/\bar{\kappa}),$$

and suppose that the parameter \bar{p} in Algorithm 3.1 is sufficiently large so that for any \bar{p} unit vectors in \mathbb{R}^n , there are two vectors such that the angle between them is at most $\bar{\alpha}$.

(a) For any $\epsilon > 0$, there is an $I > 0$ such that if $I < i < k$ and $\angle a_j^{(l)} 0 a_k^{(l_k^*)} \leq \bar{\alpha}$ for some $(j, l) \in S_{k-1}$, then

$$\frac{\|x_k - \bar{x}\|}{\|x_i - \bar{x}\|} \leq \frac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|} \leq \epsilon.$$

(b) Suppose in addition all the sets K_l have the SOSH property at \bar{x} . Then there are $M > 0$ and $I > 0$ such that if $I < i < k$ and $\angle a_j^{(l)} 0 a_k^{(l_k^*)} \leq \bar{\alpha}$ for some $(j, l) \in S_{k-1}$, then

$$\frac{\|x_k - \bar{x}\|}{\|x_i - \bar{x}\|^2} \leq \frac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|^2} < M. \quad (3.8)$$

Proof. We break up into two steps:

Step 1: Summary of results largely unchanged from [Pan14b].

We summarize the results proved for the algorithm in [Pan14b] that still hold for Algorithm 3.1 with minor modifications.

Since condition (1) holds, there is a constant $\bar{\kappa}$ satisfying the local metric inequality (3.7). It was proved in [Pan14b] that if condition (1) is satisfied, then

$$\lim_{i \rightarrow \infty} d\left(\frac{x_i - \bar{x}}{\|x_i - \bar{x}\|}, N_K(\bar{x})\right) = 0. \quad (3.9)$$

The conclusion (3.9) still holds true for Algorithm 3.1 with exactly the same proof, using only the weaker requirements of property (3) of step 2 of Algorithm 3.1 and the outer semicontinuity of the normal mapping of a convex set. The formula (3.9) is the most tedious result in [Pan14b]. With the same steps as presented in [Pan14b], we can use (3.9) to prove that $\lim_{i \rightarrow \infty} \frac{d(x_i, K)}{\|x_i - \bar{x}\|} = 1$. In view of the local metric inequality (3.7), we can consider any $\kappa > \bar{\kappa}$ and get

$$\|x_i - \bar{x}\| \leq \kappa \max_{1 \leq l \leq r} d(x_i, K_l) \text{ for all } i \text{ large enough.} \quad (3.10)$$

Step 2: Obtaining conclusions

Consider the two dimensional affine space $\bar{x} + \text{span}\{a_j^{(l)}, a_k^{(l_k^*)}\}$. Let $x_k^+ := P_{K_{l_k^*}}(x_k)$. Let the projection of x_k and x_k^+ onto this subspace be Πx_k and Πx_k^+ . Let x_k^{++} be the projection of x_k onto the hyperplane passing through \bar{x} whose normal is $a_k^{(l_k^*)}$, and let Πx_k^{++} be similarly defined. The points are indicated in Figure 3.1.

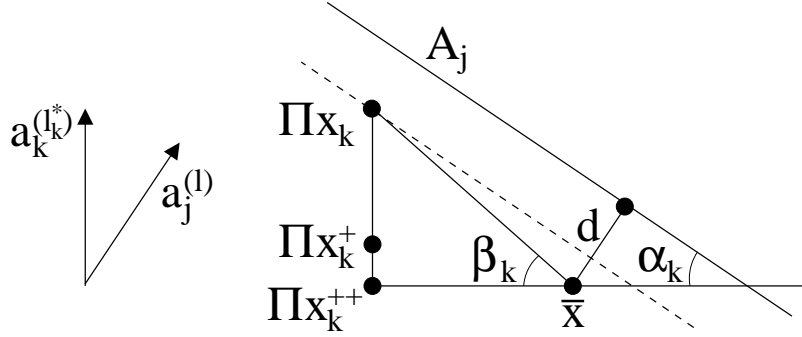


FIGURE 3.1. We illustrate the points defined in step 2 of the proof of Theorem 3.10. The directions (and not magnitudes) of $a_j^{(l)}$ and $a_k^{(l*)}$ (which are the normal vectors of the halfspaces obtained from projecting x_j onto K_l and x_k onto $K_{l_k^*}$ respectively) are indicated on the left. The distance d equals $d(\bar{x}, A_j)$. By pulling the hyperplane A_j towards \bar{x} till it hits Πx_k (the dashed line), we can prove inequality (3.13).

It is clear that $\|x_k - x_k^+\| = \|x_k - P_{K_{l_k^*}}(x_k)\| = d(x_k, K_{l_k^*})$, so from (3.10), we have, for all k large enough,

$$\begin{aligned} \|\Pi x_k - \bar{x}\| &\leq \|x_k - \bar{x}\| \\ &\leq \kappa \|x_k - P_{K_{l_k^*}}(x_k)\| \\ &= \kappa \|\Pi x_k - \Pi x_k^+\| \\ &\leq \kappa \|\Pi x_k - \Pi x_k^{++}\|. \end{aligned} \quad (3.11)$$

Hence the angle $\angle(\Pi x_k)\bar{x}(\Pi x_k^{++})$, which is marked as β_k in Figure 3.1, satisfies $\beta_k = \sin^{-1}(1/\kappa)$. Let $\bar{\beta} := \liminf_{k \rightarrow \infty} \beta_k$. We must have $\bar{\beta} \geq \sin^{-1}(1/\bar{\kappa})$. The angle α_k marked on Figure 3.1 equals $\angle a_j^{(l)} 0 a_k^{(l*)}$, and satisfies $\alpha_k \leq \bar{\alpha}$ by the assumptions in this result. The κ can be chosen such that $\sin \bar{\alpha} < \frac{1}{\kappa} < \frac{1}{\bar{\kappa}}$ so that $\bar{\alpha} < \bar{\beta}$.

Now, let A_j be the hyperplane produced by projecting x_j onto K_l (i.e., $A_j = \partial H_{(j,l)}^{\leq}$, the boundary of $H_{(j,l)}^{\leq}$), and let d_j be the distance $d(\bar{x}, A_j)$. In view of Theorem 3.4, for any $\epsilon > 0$, we can find I large enough such that

$$d(\bar{x}, A_j) \leq \epsilon \|P_{K_l}(x_j) - \bar{x}\| \leq \epsilon \|x_j - \bar{x}\| \text{ for all } j > I. \quad (3.12)$$

A simple argument in plane geometry elaborated in Figure 3.1 gives, for all $j > I$,

$$\|\Pi x_k - \Pi x_k^{++}\| \leq \frac{d(\bar{x}, A_j)}{\sin^{-1}(\beta_k - \alpha_k)} \leq \frac{\epsilon \|x_j - \bar{x}\|}{\sin^{-1}(\beta_k - \alpha_k)}. \quad (3.13)$$

Combining inequalities (3.11) and (3.13), we have, for $k > j > i > I$,

$$\begin{aligned} \|x_k - \bar{x}\| &\leq \kappa \|\Pi x_k - \Pi x_k^{++}\| \\ &\leq \kappa \frac{\epsilon \|x_j - \bar{x}\|}{\sin^{-1}(\beta_k - \alpha_k)} \\ \Rightarrow \frac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|} &\leq \frac{\kappa \epsilon}{\sin^{-1}(\bar{\beta} - \bar{\alpha})}. \end{aligned}$$

In the case where K_l have the SOSH property at \bar{x} , we can replace the $\epsilon\|x_j - \bar{x}\|$ in (3.12) by $M\|x_j - \bar{x}\|^2$. Reworking through the inequalities gives

$$\frac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|^2} \leq \frac{\kappa M}{\sin^{-1}(\bar{\beta} - \bar{\alpha})} \text{ if } k > j > i > I.$$

In view of the fact that $\|x_j - \bar{x}\| \leq \|x_i - \bar{x}\|$ from condition (2) of Step 2 of Algorithm 3.1, we have the result we need. \square

We list a few corollaries that are straightforward from Theorem 3.10. The \bar{p} -term superlinear convergence in Corollary 3.11 was the original conclusion in [Pan14b]. Corollary 3.12 shows that the convergence can be much faster for smooth problems.

Corollary 3.11. (*\bar{p} -term superlinear convergence for SIP*) Suppose Algorithm 3.1 with parameter \bar{p} produces a sequence $\{x_i\}$ that converges to a point $\bar{x} \in K$, and the convergence is not finite, and condition (1) of Theorem 3.10 holds.

From condition (1) and Lemma 3.9, $\{K_l\}_{l=1}^r$ satisfies the local metric inequality at \bar{x} with some constant, say $\bar{\kappa}$. Let $\bar{\alpha}$ be such that

$$\bar{\alpha} < \sin^{-1}(1/\bar{\kappa}),$$

and suppose that the parameter \bar{p} in Algorithm 3.1 is sufficiently large so that for any \bar{p} unit vectors in \mathbb{R}^n , there are two vectors such that the angle between them is at most $\bar{\alpha}$. Suppose also that $S_i = \{\max(1, i - \bar{p}), \dots, i\} \times \{1, \dots, r\}$ (which implies that S_i has $(\bar{p} + 1)r$ elements). Then $\{x_i\}$ converge \bar{p} -term superlinearly to \bar{x} , i.e.,

$$\lim_{i \rightarrow \infty} \frac{\|x_{i+\bar{p}} - \bar{x}\|}{\|x_i - \bar{x}\|} = 0. \quad (3.14)$$

If in addition the sets K_l have the SOSH property at \bar{x} for all $l \in \{1, \dots, r\}$, then $\{x_i\}$ converge \bar{p} -term quadratically to \bar{x} , i.e.,

$$\limsup_{i \rightarrow \infty} \frac{\|x_{i+\bar{p}} - \bar{x}\|}{\|x_i - \bar{x}\|^2} < \infty. \quad (3.15)$$

Corollary 3.12. (*Fast convergence for smooth SIP*) Suppose Algorithm 3.1 with parameter \bar{p} is such that $S_i = \{i\} \times \{1, \dots, r\}$ (which implies that S_i has r elements) produces a sequence $\{x_i\}$ that converges to a point $\bar{x} \in K$, and the convergence is not finite, and condition (1) of Theorem 3.10 holds. If $N_{K_l}(\bar{x})$ contains only one nonzero direction for all $l \in \{1, \dots, r\}$, then the convergence of $\{x_i\}$ to \bar{x} is superlinear, i.e., $\lim_{i \rightarrow \infty} \frac{\|x_{i+1} - \bar{x}\|}{\|x_i - \bar{x}\|} = 0$. If in addition the sets K_l have the SOSH property at \bar{x} , then the convergence of $\{x_i\}$ to \bar{x} is quadratic, i.e., $\limsup_{i \rightarrow \infty} \frac{\|x_{i+1} - \bar{x}\|}{\|x_i - \bar{x}\|^2} < \infty$.

Proof. Due to the fact that the graph of the normal cone mapping $N_C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is closed for any closed convex set $C \subset \mathbb{R}^n$, the unit vectors of the normals obtained by projecting x_i onto each K_l converge to the only direction of unit length in $N_{K_l}(\bar{x})$ for $l \in \{1, \dots, r\}$.

We now examine Statement (a) of Theorem 3.10. Choose any $\bar{\alpha} > 0$. As a consequence of the outer semicontinuity of the mapping $N_{K_l} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ at \bar{x} for all $l \in \{1, \dots, r\}$, there is an I_1 such that if $i > I_1$, then $\angle a_i^{(l_{i+1}^*)} 0a_{i+1}^{(l_{i+1}^*)} < \bar{\alpha}$. Hence for any $\epsilon > 0$, we can increase I_1 if necessary so that if $i > I_1$, then $\frac{\|x_{i+1} - \bar{x}\|}{\|x_i - \bar{x}\|} \leq \epsilon$. A similar conclusion holds for quadratic convergence. \square

We make another remark about higher order \bar{p} -term convergence.

Remark 3.13. (Higher order \bar{p} -term convergence) We note that \bar{p} -term quadratic convergence (3.15) implies

$$\limsup_{i \rightarrow \infty} \frac{\|x_{i+2\bar{p}} - \bar{x}\|}{\|x_i - \bar{x}\|^4} < \infty, \quad (3.16)$$

which would be $(2\bar{p})$ -term quartic convergence. So if we do not have bounds for the parameter \bar{p} , then the degree of the denominator of the term in (3.16) can be set arbitrarily high, and the limit superior can also taken to be zero. It is therefore important to bound \bar{p} .

Remark 3.14. (Algorithmic consequences of Theorem 3.10) An insight obtained from Theorem 3.10 for how an algorithm might run in practice is that after a few iterations, the unit normal vectors of the halfspaces generated by the projection process can be compared. If a unit normal vector of an old halfspace is close enough to a newer one, then the old halfspace can be removed for the next QP subproblem of projecting onto a polyhedron.

4. A SUBGRADIENT ALGORITHM FOR THE CIP

In this section, we show how the ideas for the SIP can be transferred to the CIP. We write down the SGQP (subgradient quadratic programming) algorithm for solving the CIP (1.2).

Algorithm 4.1. (SGQP algorithm for the CIP) For a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and starting iterate x_0 , we find a point \bar{x} such that $f(\bar{x}) \leq 0$.

Step 0: Set $i = 0$ and let \bar{p} be a fixed positive integer.

Step 1: Find $y_i \in \partial f(x_i)$ and define the halfspace $H_i^\leq \subset \mathbb{R}^n$ by

$$H_i^\leq := \{x \in \mathbb{R}^n \mid f(x_i) + \langle y_i, x - x_i \rangle \leq 0\}. \quad (4.1)$$

Step 2: Find x_{i+1} such that

(1) $x_{i+1} \in F_i$, where S_i is a subset of $\{\max(i-\bar{p}, 0), \dots, i\}$ and $F_i := \bigcap_{k \in S_i} H_k^\leq$.

(2) $\|x_{i+1} - c\| \leq \|x_i - c\|$ for all $c \in f^{-1}((-\infty, 0])$.

(3) $x_i - x_{i+1}$ lies in $\text{conv}\{y_k\}_{k \in S_i}$, i.e., $x_i - x_{i+1}$ lies in the cone generated by the convex hull of the normals of the halfspaces $\{H_k^\leq\}_{k \in S_i}$.

Step 3: Set $i \leftarrow i + 1$, and go back to step 1.

When $S_i \equiv \{i\}$ in Algorithm 4.1, the iterate x_{i+1} can be calculated to be $x_{i+1} = x_i - \frac{f(x_i)}{\|y_i\|^2} y_i$. It is easy to check that Algorithm 4.1 converges linearly when $S_i \equiv \{i\}$ for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x_1, x_2) = \max(2x_1 - x_2, 2x_2 - x_1)$ exhibits the zigzagging behavior typical of alternating projections, but converges in finitely many iterations when $S_i \supset \{i, i-1\}$ for large i .

Remark 3.3 also applies to Step 2 of Algorithm 4.1; We can take x_{i+1} to be the projection of x_i onto F_i in Step 2 of 4.1.

Theorem 4.2. (Basic convergence for the CIP) Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, and $f^{-1}((-\infty, 0]) \neq \emptyset$. Then Algorithm 4.1 for any parameter \bar{p} and $S_i \supset \{i\}$ converges to a point \bar{x} such that $f(\bar{x}) \leq 0$.

Proof. By Theorem 2.3(1), the sequence of iterates $\{x_i\}$ is bounded, and has a convergent subsequence. Suppose it has a cluster point \bar{x} . Seeking a contradiction, suppose $f(\bar{x}) > 0$. Since $0 \notin \partial f(\bar{x})$, let $\gamma := \sup\{\|y\| : y \in \partial f(\bar{x})\}$. If x_i is sufficiently close to \bar{x} , then for the choice $y_i \in \partial f(x_i)$, the distance of x_i to the halfspace H_i^\leq

defined in (4.1) is $f(x_i)/\|y_i\|$. By the outer semicontinuity of the subdifferential and the continuity of $f(\cdot)$, the value $f(x_i)/\|y_i\|$ is in turn bounded from below by $\frac{f(\bar{x})}{2\gamma}$ if x_i is sufficiently close to \bar{x} . This implies that

$$\|x_i - x_{i+1}\| \geq \frac{f(\bar{x})}{2\gamma}. \quad (4.2)$$

We simplify the statements in the proof by letting C to be $f^{-1}((-\infty, 0])$. Consider iterates x_i and x_{i+1} . Since x_{i+1} is the projection of x_i onto a set containing C , we have $\langle x_i - x_{i+1}, P_C(x_i) - x_{i+1} \rangle \leq 0$. This inequality implies that

$$\|x_i - x_{i+1}\|^2 + \|x_{i+1} - P_C(x_i)\|^2 \leq \|x_i - P_C(x_i)\|^2,$$

which in turn gives

$$\begin{aligned} d(x_{i+1}, C)^2 &\leq \|x_{i+1} - P_C(x_i)\|^2 \\ &\leq \|x_i - P_C(x_i)\|^2 - \|x_i - x_{i+1}\|^2 \\ &= d(x_i, C)^2 - \|x_i - x_{i+1}\|^2. \end{aligned}$$

It follows from the continuity of $d(\cdot, C)$ and (4.2) that if x_i were sufficiently close to \bar{x} , then $d(x_{i+1}, C)^2 \leq d(\bar{x}, C)^2 - [\frac{f(\bar{x})}{2\gamma}]^2$. This fact and Theorem 2.3(1) contradicts the assumption that \bar{x} is a cluster point of $\{x_i\}$. Therefore, the cluster points of $\{x_i\}$ must belong to C . By Theorem 2.3(2), we conclude that $\{x_i\}$ converges to some point \bar{x} in C . In other words, $\{x_i\}$ converges to some \bar{x} such that $f(\bar{x}) \leq 0$. \square

We now prove a few intermediate inequalities useful for Theorem 4.4.

Lemma 4.3. (*Intermediate inequalities*) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function. Choose $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) = 0$ and $0 \notin \partial f(\bar{x})$. Let γ_1 be such that $\gamma_1 < d(0, \partial f(\bar{x}))$ and $\gamma_2 < \frac{d(0, \partial f(\bar{x}))}{\sup_{y \in \partial f(\bar{x})} \|y\|}$. Then there is some $\epsilon > 0$ such that for all x such that $\|x - \bar{x}\| < \epsilon$, $f(x) > 0$ and $d(\frac{x - \bar{x}}{\|x - \bar{x}\|}, N_{f^{-1}((-\infty, 0])}(\bar{x})) < \epsilon$, we have

- (1) $\frac{f(x)}{\|x - \bar{x}\|} \geq \gamma_1$.
- (2) $\left\langle \frac{y}{\|y\|}, \frac{x - \bar{x}}{\|x - \bar{x}\|} \right\rangle > \gamma_2$ for all $y \in \partial f(x)$.

Proof. By the convexity of $f(\cdot)$, we have

$$\begin{aligned} f(x) &\geq f(\bar{x}) + \sup_{z \in \partial f(\bar{x})} \langle z, x - \bar{x} \rangle \\ \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|} &\geq \sup_{z \in \partial f(\bar{x})} \left\langle z, \frac{x - \bar{x}}{\|x - \bar{x}\|} \right\rangle. \end{aligned} \quad (4.3)$$

Since $d(\frac{x - \bar{x}}{\|x - \bar{x}\|}, N_{f^{-1}((-\infty, 0])}(\bar{x}) \cap \partial \mathbb{B}) < \epsilon$, we can find v such that $\|v\| = 1$, $\|v - \frac{x - \bar{x}}{\|x - \bar{x}\|}\| < \epsilon$ and $v \in N_{f^{-1}((-\infty, 0])}(\bar{x})$. Let $w = v - \frac{x - \bar{x}}{\|x - \bar{x}\|}$. We now make use of the well known fact that $\text{lip } f(\bar{x}) = \max_{z \in \partial f(\bar{x})} \|z\|$. For any $z \in \partial f(\bar{x})$, we have

$$-\langle z, w \rangle \geq -\|w\| \text{lip } f(\bar{x}) \geq -\epsilon \text{lip } f(\bar{x}).$$

Since $N_{f^{-1}((-\infty, 0])}(\bar{x}) = \text{pos}(\partial f(\bar{x}))$ by [RW98, Proposition 10.3], there is some $z' \in \partial f(\bar{x})$ such that $v = \frac{z'}{\|z'\|}$. We have

$$\begin{aligned}
 \sup_{z \in \partial f(\bar{x})} \left\langle z, \frac{x - \bar{x}}{\|x - \bar{x}\|} \right\rangle &= \sup_{z \in \partial f(\bar{x})} [\langle z, v \rangle - \langle z, w \rangle] \\
 &\geq \sup_{z \in \partial f(\bar{x})} [\langle z, v \rangle - \epsilon \text{lip } f(\bar{x})] \\
 &\geq \frac{\langle z', z' \rangle}{\|z'\|} - \epsilon \text{lip } f(\bar{x}) \\
 &= \|z'\| - \epsilon \text{lip } f(\bar{x}) \\
 &\geq d(0, \partial f(\bar{x})) - \epsilon \text{lip } f(\bar{x}). \tag{4.4}
 \end{aligned}$$

Since $0 \notin \partial f(\bar{x})$, $d(0, \partial f(\bar{x})) > 0$. We can reduce $\epsilon > 0$ if necessary, so by combining (4.3) and (4.4), we have conclusion (1).

Let $y \in \partial f(x)$. We have

$$\begin{aligned}
 f(\bar{x}) &\geq f(x) + \langle y, \bar{x} - x \rangle \\
 \Rightarrow \left\langle \frac{y}{\|y\|}, \frac{x - \bar{x}}{\|x - \bar{x}\|} \right\rangle &\geq \frac{f(x)}{\|y\| \|x - \bar{x}\|}. \tag{4.5}
 \end{aligned}$$

The outer semicontinuity of $\partial f(\cdot)$ implies that for any $\delta > 0$, we can reduce ϵ if necessary so that $\|y\| \leq \sup_{y' \in \partial f(\bar{x})} \|y'\| + \delta$ whenever $y \in \partial f(x)$ and $\|x - \bar{x}\| \leq \epsilon$. By combining this observation to (4.5) together with conclusion (1), we get conclusion (2). \square

We present our result on the convergence of the CIP.

Theorem 4.4. (Convergence rates for the CIP) Suppose that Algorithm 4.1 with parameter \bar{p} for a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ produces a sequence $\{x_i\}$ that converges to a point $\bar{x} \in f^{-1}(0)$ such that $0 \notin \partial f(\bar{x})$, and the convergence to \bar{x} is not finite. Let $\bar{\alpha} > 0$ be such that

$$\bar{\alpha} < \sin^{-1} \left(\frac{d(0, \partial f(\bar{x}))}{\sup_{y \in \partial f(\bar{x})} \|y\|} \right).$$

Suppose that the \bar{p} in Algorithm 4.1 is sufficiently large so that for any \bar{p} unit vectors in \mathbb{R}^n , there are two vectors such that the angle between them is at most $\bar{\alpha}$.

(a) For any $\epsilon > 0$, there is an $I > 0$ such that if $I < i < k$ and $\angle y_j 0 y_k \leq \bar{\alpha}$ for some $j \in \{i, i-1, \dots, k-1\} \cap S_k$, then

$$\frac{\|x_k - \bar{x}\|}{\|x_i - \bar{x}\|} \leq \frac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|} \leq \epsilon.$$

(b) If in addition f is strongly semismooth at \bar{x} , then there are $M > 0$ and $I > 0$ such that if $I < i < k$ and $\angle y_j 0 y_k \leq \bar{\alpha}$ for some $j \in \{i, i-1, \dots, k-1\} \cap S_k$, then

$$\frac{\|x_k - \bar{x}\|}{\|x_i - \bar{x}\|^2} \leq \frac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|^2} < M. \tag{4.6}$$

Proof. The proof has quite a few parts.

Part 1: Any cluster point of $\{\frac{x_i - \bar{x}}{\|x_i - \bar{x}\|}\}$ is in $N_{f^{-1}((-\infty, 0])}(\bar{x}) = \text{pos}(\partial f(\bar{x}))$.

This part of the proof contains ideas in the proof of [Pan14b, Proposition 5.8], but is much simpler. We use the variables j and k to be running variables incompatible

with (a) and (b) of the main result. By the design of step 2, part (3) of Algorithm 4.1, we have

$$x_{i+1} = x_i - \sum_{k=\max(1, i-\bar{p})}^i \lambda_{i,k} y_k,$$

where $\lambda_{i,k} \geq 0$ for all i and k such that $\max(1, i-\bar{p}) \leq k \leq i$, and $\lambda_{i,k} = 0$ if $k \notin S_i$. For $j > i$, we then have

$$x_i - x_j = \sum_{s=i}^{j-1} \sum_{k=\max(1, s-\bar{p})}^s \lambda_{s,k} y_k.$$

We have

$$x_i - \bar{x} = \lim_{j \rightarrow \infty} \sum_{s=i}^{j-1} \sum_{k=\max(1, s-\bar{p})}^s \lambda_{s,k} y_k. \quad (4.7)$$

We now show that the convergence of (4.7) is absolute so that an infinite sum notation is justified. By the outer semicontinuity of the subdifferential mapping $\partial f(\cdot)$, for any $\epsilon > 0$, we can find I large enough so that $\partial f(x_i) \subset \partial f(\bar{x}) + \epsilon \mathbb{B}$ for all $i > I - \bar{p}$. Since $0 \notin \partial f(\bar{x})$, we can choose ϵ small enough so that $0 \notin \partial f(\bar{x}) + \epsilon \mathbb{B}$. The set $\text{pos}(\partial f(\bar{x}) + \epsilon \mathbb{B})$ is a pointed cone, so by Lemma 2.4, there is a constant $m > 0$ such that

$$m \sum_{s=i}^{j-1} \sum_{k=\max(1, s-\bar{p})}^s \lambda_{s,k} \|y_k\| \leq \left\| \sum_{s=i}^{j-1} \sum_{k=\max(1, s-\bar{p})}^s \lambda_{s,k} y_k \right\| = \|x_i - x_j\|.$$

Taking the limits as $j \rightarrow \infty$, we have

$$\sum_{s=i}^{\infty} \sum_{k=\max(1, s-\bar{p})}^s \lambda_{s,k} \|y_k\| \leq \frac{1}{m} \|x_i - \bar{x}\| < \infty,$$

which shows that the convergence of (4.7) is absolute.

For any $\epsilon > 0$, we can always choose i large enough so that $y_k \in \partial f(\bar{x}) + \epsilon \mathbb{B}$ for all $k \geq i - \bar{p}$. This means that

$$x_i - \bar{x} = \sum_{s=i}^{\infty} \sum_{k=\max(1, s-\bar{p})}^s \lambda_{s,k} y_k \in \text{pos}(\partial f(\bar{x}) + \epsilon \mathbb{B}).$$

Since $\text{pos}(\partial f(\bar{x}))$ is a pointed cone, the set $\text{pos}(\partial f(\bar{x}) + \epsilon \mathbb{B}) \cap \partial \mathbb{B}$ converges to $\text{pos}(\partial f(\bar{x})) \cap \partial \mathbb{B}$ in the Pompeiu Hausdorff distance as $\epsilon \searrow 0$. So the cluster points of $\{\frac{x_i - \bar{x}}{\|x_i - \bar{x}\|}\}$ lie in $\text{pos}(\partial f(\bar{x}))$. Since $\text{pos}(\partial f(\bar{x}))$ equals $N_{f^{-1}((-\infty, 0])}(\bar{x})$ by [RW98, Proposition 10.3], we are done.

Part 2: Applying Lemma 4.3.

By applying Lemma 4.3 and part 1, we deduce that if $\gamma_1 < d(0, \partial f(\bar{x}))$ and $\gamma_2 < \frac{d(0, \partial f(\bar{x}))}{\sup_{y \in \partial f(\bar{x})} \|y\|}$, then there is an $I > 0$ such that

$$\frac{f(x_i)}{\|x_i - \bar{x}\|} > \gamma_1 \quad (4.8a)$$

$$\text{and } \left\langle \frac{x_i - \bar{x}}{\|x_i - \bar{x}\|}, \frac{y_i}{\|y_i\|} \right\rangle > \gamma_2 \text{ for all } i > I. \quad (4.8b)$$

Part 3: Defining and evaluating $\limsup_{j \rightarrow \infty} \lambda_j$ and $\limsup_{j \rightarrow \infty} \frac{\lambda_j}{\|x - x_j\|}$.

Define $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$ to be $f_j(x) := f(x_j) + \langle y_j, x - x_j \rangle$. In view of (4.8a), we have $f_j(x_j) = f(x_j) \geq \gamma_1 \|\bar{x} - x_j\|$ for j large enough. Let $\lambda_j \in \mathbb{R}$ be such that $f_j(\lambda_j(x_j - \bar{x}) + \bar{x}) = 0$. By the convexity of $f(\cdot)$, it is clear that $\lambda_j \geq 0$. Since $f(x_j) > 0$, we have $\lambda_j < 1$. By the semismoothness of f at \bar{x} and applying (4.8a), we have

$$\begin{aligned}
 (1 - \lambda_j)f_j(\bar{x}) + \lambda_j f_j(x_j) &= f_j(\lambda_j(x_j - \bar{x}) + \bar{x}) = 0 \\
 \Rightarrow \lambda_j &= \frac{-f_j(\bar{x})}{f_j(x_j) - f_j(\bar{x})} \\
 &= \frac{-[f(x_j) + \langle y_j, \bar{x} - x_j \rangle]}{f(x_j) - [f(x_j) + \langle y_j, \bar{x} - x_j \rangle]} \\
 &= \frac{f(\bar{x}) - f(x_j) - \langle y_j, \bar{x} - x_j \rangle}{f(x_j) + f(\bar{x}) - f(x_j) - \langle y_j, \bar{x} - x_j \rangle} \\
 &\leq \frac{o(\|\bar{x} - x_j\|)}{\gamma_1 \|\bar{x} - x_j\| + o(\|\bar{x} - x_j\|)} \\
 &= o(1).
 \end{aligned} \tag{4.9}$$

In other words, $\limsup_{j \rightarrow \infty} \lambda_j = 0$. In the case where f is strongly semismooth at \bar{x} , we can repeat the calculations to get

$$\lambda_j \leq \frac{O(\|\bar{x} - x_j\|^2)}{\gamma_1 \|\bar{x} - x_j\| + O(\|\bar{x} - x_j\|^2)} = O(\|\bar{x} - x_j\|),$$

or $\limsup_{j \rightarrow \infty} \frac{\lambda_j}{\|\bar{x} - x_j\|} < \infty$.

Part 4: Bounding $\frac{\|x_k - \bar{x}\|}{\|x_k - x_k^+\|}$.

We shall let x_k^+ be the point $x_k - \frac{f(x_k)}{\|y_k\|^2} y_k$. The point x_k^+ is also the projection of x_k onto H_k^\leq . The distance $\|x_k - x_k^+\|$ is easily calculated to be $\frac{|f(x_k)|}{\|y_k\|}$. The distance from x_k to the hyperplane with normal y_k passing through \bar{x} can be calculated to be $\frac{1}{1 - \lambda_k} \frac{|f(x_k)|}{\|y_k\|}$.

We now show that for any $\gamma_2 < \frac{d(0, \partial f(\bar{x}))}{\sup_{y \in \partial f(\bar{x})} \|y\|}$, we can find I such that if $k > I$, then the angle

$$\theta_k := \angle x_k^+ x_k \bar{x} \tag{4.10}$$

is such that $\theta_k \leq \cos^{-1} \gamma_2$ by (4.8b). Since $x_k - x_k^+$ is a positive multiple of y_k , by (4.8a),

$$\cos \theta_k = \left\langle \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|}, \frac{y_k}{\|y_k\|} \right\rangle > \gamma_2,$$

which gives us what we need.

Now

$$\begin{aligned}
 (1 - \lambda_k) \gamma_2 \|x_k - \bar{x}\| &\leq (1 - \lambda_k) \|x_k - \bar{x}\| \cos \theta_k \\
 &= [1 - \lambda_k] \frac{1}{1 - \lambda_k} \frac{|f(x_k)|}{\|y_k\|} \\
 &= \|x_k - x_k^+\|.
 \end{aligned}$$

Therefore, if k is large enough, we have

$$\|x_k - \bar{x}\| \leq K \|x_k - x_k^+\|, \tag{4.11}$$

where $K = 1.1/\gamma_2$.

Part 5: Evaluating $\frac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|}$ and $\frac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|^2}$, and wrapping up.

We consider the two-dimensional space containing \bar{x} , $\bar{x} + y_j$ and $\bar{x} + y_k$. Let x_k^{++} be the point on the line containing Πx_k and Πx_k^+ such that $\angle(\Pi x_k)\bar{x}(\Pi x_k^{++}) = \pi/2$. We shall call the projections of x_k , x_k^+ and x_k^{++} onto this 2d space to be Πx_k , Πx_k^+ and Πx_k^{++} . We note that the distance from \bar{x} to $A_j := \partial H_j^{\leq}$, the boundary of H_j^{\leq} , is bounded above by $\lambda_j \|x_j - \bar{x}\|$ because $\lambda_j(x_j - \bar{x}) + \bar{x} \in A_j$ and $\|[\lambda_j(x_j - \bar{x}) + \bar{x}] - \bar{x}\| = \lambda_j \|x_j - \bar{x}\|$. We can refer back to Figure 3.1, but replace $a_k^{(l_k^*)}$ by y_k and $a_j^{(l_j)}$ by y_j .

Making use of (1) in step 2 of Algorithm 4.1, we can use elementary geometry (in the same manner as in the proof of (3.13)) to prove the inequality

$$\|\Pi x_k - \bar{x}\| \leq \frac{d(\bar{x}, A_j)}{\sin(\beta_k - \alpha_k)} \leq \frac{\lambda_j \|x_j - \bar{x}\|}{\sin(\beta_k - \alpha_k)}, \quad (4.12)$$

where $\beta_k := \angle(\Pi x_k)\bar{x}(\Pi x_k^{++})$ and $\alpha_k = \angle y_j 0 y_k$. The angle β_k is bounded from below by

$$\beta_k = \angle(\Pi x_k)\bar{x}(\Pi x_k^{++}) \geq \angle x_k \bar{x} x_k^{++} = \frac{\pi}{2} - \angle x_k^+ x_k \bar{x} = \frac{\pi}{2} - \theta_k,$$

where θ_k was defined in (4.10), and is bounded from above by $\cos^{-1} \gamma_2$ for large k . For any choice of $\bar{\alpha}$, we can make γ_2 close enough to $\frac{d(0, \partial f(\bar{x}))}{\sup_{y \in \partial f(\bar{x})} \|y\|}$ so that for all i large enough, we have

$$\beta_k \geq \bar{\beta} > \bar{\alpha},$$

where $\bar{\beta} := \sin^{-1} \gamma_2$. Combining with the assumption that $\alpha_k \leq \bar{\alpha}$, (4.12) gives

$$\|\Pi x_k - \bar{x}\| \leq \frac{\lambda_j \|x_j - \bar{x}\|}{\sin(\bar{\beta} - \bar{\alpha})} \text{ for all } i \text{ large enough.} \quad (4.13)$$

We have, by (4.11) and (4.13),

$$\begin{aligned} \|x_k - \bar{x}\| &\leq K \|x_k - x_k^+\| \\ &= K \|\Pi x_k - \Pi x_k^+\| \\ &\leq K \|\Pi x_k - \bar{x}\| \\ &\leq K \frac{\lambda_j \|x_j - \bar{x}\|}{\sin(\bar{\beta} - \bar{\alpha})} \\ \Rightarrow \frac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|} &\leq \frac{K \lambda_j}{\sin(\bar{\beta} - \bar{\alpha})}. \end{aligned}$$

Making use of the fact that $\lambda_i \searrow 0$ in (4.9), for any $\epsilon > 0$, we can choose I large enough so that $\frac{K \lambda_i}{\sin(\bar{\beta} - \bar{\alpha})} < \epsilon$ for all $i > I$. Thus the conclusion that we seek holds. In the case where $f(\cdot)$ is strongly semismooth, we can make use of the fact that $\limsup_{i \rightarrow \infty} \frac{\lambda_i}{\|x_i - \bar{x}\|}$ is finite in Part 3, say with value M_1 , to get, for all j, k large enough, that

$$\frac{\|x_k - \bar{x}\|}{\|x_j - \bar{x}\|^2} \leq \frac{K(M_1 + 1)}{\sin(\bar{\beta} - \bar{\alpha})}.$$

In view of (2) of Step 2 in Algorithm 4.1, we have $\|x_j - \bar{x}\| \leq \|x_i - \bar{x}\|$. Thus our claim is proved. \square

Remark 4.5. (Applicability of Theorem 4.4 to SIP) The SIP (1.1) is a special case of the CIP (1.2), since the problem of finding a point in the intersection of r closed convex sets K_1, \dots, K_r can be regarded as the problem of finding a point x such that $f(x) := \max_{l=1}^r d(x, K_l) \leq 0$. So at first glance, Theorem 4.4 seems to be applicable

for the SIP, which would be stronger than the main result in [Pan14b]. However, the condition $0 \notin \partial f(\bar{x})$ in Theorem 4.4 is violated for the SIP.

When the parameter \bar{p} is set to 0 in Algorithm 4.1, we get linear convergence as shown below.

Theorem 4.6. (*Linear convergence for the CIP*) Suppose that Algorithm 4.1 for a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $S_i \equiv \{i\}$ for all i produces a sequence $\{x_i\}$ that converges to a point $\bar{x} \in f^{-1}(0)$ such that $0 \notin \partial f(\bar{x})$. Then the convergence is at least linear.

Proof. Note that this result is already a consequence of [Rob76], but we include its proof here since it is quite easy.

The first 2 parts of the proof of Theorem 4.4 still apply. Next, we have $x_{i+1} = x_i - \frac{f(x_i)}{\|y_i\|^2} y_i$. Using (4.8a), we get

$$\left\| \frac{f(x_i)}{\|y_i\|^2} y_i \right\| = \frac{|f(x_i)|}{\|y_i\|} \geq \frac{\gamma_1}{\max_{y \in \partial f(x_i)} \|y\|} \|x_i - \bar{x}\|. \quad (4.14)$$

By the outer semicontinuity of $\partial f(\cdot)$, there is some I such that

$$\frac{\gamma_1}{\max_{y \in \partial f(x_i)} \|y\|} > \frac{\gamma_1}{2 \max_{y \in \partial f(\bar{x})} \|y\|}, \quad (4.15)$$

and the constant on the right is positive. Also, by the property of projections, the angle $\angle x_i x_{i+1} \bar{x}$ is obtuse. Together with (4.14) and (4.15), we have

$$\begin{aligned} \|x_{i+1} - \bar{x}\|^2 &\leq \|x_i - \bar{x}\|^2 - \|x_i - x_{i+1}\|^2 \\ &= \|x_i - \bar{x}\|^2 - \left[\frac{|f(x_i)|}{\|y_i\|} \right]^2 \\ &\leq \|x_i - \bar{x}\|^2 - \left[\frac{\gamma_1}{2 \max_{y \in \partial f(\bar{x})} \|y\|} \right]^2 \|x_i - \bar{x}\|^2 \\ \frac{\|x_{i+1} - \bar{x}\|}{\|x_i - \bar{x}\|} &\leq \sqrt{1 - \left[\frac{\gamma_1}{2 \max_{y \in \partial f(\bar{x})} \|y\|} \right]^2} < 1. \end{aligned}$$

This shows that $\limsup_{i \rightarrow \infty} \frac{\|x_{i+1} - \bar{x}\|}{\|x_i - \bar{x}\|} < 1$, which is the linear convergence we seek. \square

Theorems 4.4 and 4.6 suggest that as the parameter \bar{p} increases, the constant of linear convergence gets lower, from linear convergence for the case of $\bar{p} = 0$ in Theorem 4.6 to the case of superlinear convergence in Theorem 4.4.

We show that the condition $0 \notin \partial f(\bar{x})$ cannot be dropped in Theorems 4.4 and 4.6.

Example 4.7. (Condition $0 \notin \partial f(\bar{x})$ essential in Theorems 4.4 and 4.6) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function such that

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x \in (0, 0.5] \\ 0 & \text{if } x = 0 \\ f(-x) & \text{if } x \in [-0.5, 0). \end{cases}$$

The function f is convex on $[0, 0.5]$ because $f''(x) = e^{-1/x} \frac{1-2x}{x^4}$ if $x \in (0, 0.5)$, and is positive in that region. It is clear that $f^{-1}((-\infty, 0]) = \{0\}$. Consider Algorithm 4.1

for any parameter of \bar{p} . If any iterate x_i is in $(0, 0.5)$, then the next iterate x_{i+1} is calculated to be

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{e^{-1/x_i}}{e^{-1/x_i}(1/x_i^2)} = x_i - x_i^2.$$

It is clear that $\lim_{i \rightarrow \infty} x_i = 0$, and that $\lim_{i \rightarrow \infty} \frac{|x_{i+1}-0|}{|x_i-0|} = 1$, so there is no linear convergence.

Robinson [Rob76] included his reasons for analyzing the KCIP (1.3) for only the case when the domain of f is \mathbb{R}^n and not a Hilbert space in general, and these arguments carry over to the main results here as well.

Remark 4.8. (On CIP involving many smooth convex functions) The original CIP studied in [GP98, GP01] was, for convex functions $f_l : \mathbb{R}^n \rightarrow \mathbb{R}$ defined for $l \in \{1, \dots, r\}$,

$$\text{Find } x \in \mathbb{R}^n \text{ s.t. } f_l(x) \leq 0 \text{ for all } l \in \{1, \dots, r\}. \quad (4.16)$$

One can treat the problem above as a CIP in our setting by considering

$$f_{\max}(\cdot) := \max_{l \in \{1, \dots, r\}} f_l(\cdot). \quad (4.17)$$

A natural question to ask is whether the analysis in this section can be generalized if we had studied (4.16) instead. Unfortunately, Lemma 4.3 cannot be easily extended. For example, consider $f_l : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f_l(x) = x_l$ (i.e., taking the l th coordinate). There is no constant $\gamma_1 > 0$ such that for $l \in \{1, 2\}$, if $f_l(x) > 0$, then $f_l(x) \geq \gamma_1 \|x\|$. Such a constant has to exist in order for Part 3 of the proof of Theorem 4.4 to go through.

If $f_l(\cdot)$ were smooth convex functions for $l \in \{1, \dots, r\}$, we can analyze f_{\max} in (4.17) using the results in this section to obtain r -term superlinear or r -term quadratic convergence (defined in (3.14) and (3.15)) to a point $\bar{x} \in f_{\max}^{-1}((-\infty, 0])$ if the subdifferential of f_{\max} at an iterate x_i is taken to be $\nabla f_l(x_i)$ for some $l \in \{1, \dots, r\}$.

5. INFEASIBILITY

Closely related to the SIP is the *Best Approximation Problem* (BAP): For a Hilbert space X , a point $x_0 \in X$ and r closed convex sets K_i for $i = 1, \dots, r$, find the closest point to x_0 in $K := \cap K_i$. That is,

$$\begin{aligned} \text{(BAP):} \quad & \min_{x \in X} \|x - x_0\| \\ \text{s.t.} \quad & x \in K := \bigcap_{i=1}^r K_i. \end{aligned} \quad (5.1)$$

The case when the BAP is infeasible was discussed in [Pan14b, Section 6]. In this section, we discuss the case where the SIP and CIP are infeasible, and show that for the SIP and BAP, one can use a finite number of operations to find a certificate of infeasibility. We also make another observation for the BAP.

We present the algorithm for the BAP needed for future discussions.

Algorithm 5.1. (*BAP algorithm*) For a point x_0 and closed convex sets K_l , $l = 1, 2, \dots, r$, of a Hilbert space X , find the closest point to x_0 in $K := \cap_{l=1}^r K_l$.

Step 0: Let $i = 0$.

Step 1: For $l \in \{1, \dots, r\}$, define $x_i^{(l)} \in X$, $a_i^{(l)} \in X$ and $b_i^{(l)} \in \mathbb{R}$ by

$$\begin{aligned} x_i^{(l)} &= P_{K_l}(x_i), \\ a_i^{(l)} &= x_i - x_i^{(l)}, \\ \text{and } b_i^{(l)} &= \langle a_i^{(l)}, x_i^{(l)} \rangle. \end{aligned}$$

For each i , we let $l_i^* \in \{1, \dots, r\}$ be such that

$$l_i^* := \arg \max_{1 \leq l \leq r} \|x_i - P_{K_l}(x_i)\|.$$

Step 2: Choose $S_i \subset \{\max(i - \bar{p}, 0), \dots, i\} \times \{1, \dots, r\}$, and define $F_i \subset X$ by

$$F_i := \bigcap_{(j,l) \in S_i} H_{(j,l)}^{\leq}, \quad (5.2)$$

$$\text{where } H_{j,i}^{\leq} := \left\{ x : \langle a_j^{(l)}, x \rangle \leq b_j^{(l)} \right\}. \quad (5.3)$$

Let $x_{i+1} = P_{F_i}(x_0)$.

Step 3: Set $i \leftarrow i + 1$, and go back to step 1.

We now present our first result that will supersede [Pan14b, Theorem 6.1].

Theorem 5.2. (No strong cluster points under infeasibility) Let X be a Hilbert space, $x_0 \in \mathbb{R}^n$, and let $K_l \subset X$ be closed convex sets for $l = 1, \dots, r$. Suppose $K := \bigcap_{l=1}^r K_l$ is the empty set. Consider the following scenarios for solving the SIP, CIP and BAP respectively.

- (1) Suppose Algorithm 3.1 is run for the SIP with $\bar{p} = \infty$ at step 0 and the conditions

$$S_i \subset S_{i+1} \text{ for all } i \geq 0, \quad (5.4a)$$

$$\text{and } (i, l_i^*) \in S_i \text{ for all } i \geq 0 \quad (5.4b)$$

hold. If $\tilde{F}_i = \emptyset$ for some i , then infeasibility is detected. Otherwise, the sequence $\{x_i\}$ produced cannot have strong cluster points.

- (2) Suppose Algorithm 4.1 is run for the CIP with $\bar{p} = \infty$ in step 0, and $S_i = \{0, 1, \dots, i\}$ for all $i \geq 0$. If $F_i = \emptyset$ for some i , then infeasibility is detected. Otherwise, the sequence $\{x_i\}$ produced cannot have strong cluster points.
- (3) Suppose Algorithm 5.1 is run for the BAP, and (5.4) holds. If $F_i = \emptyset$ for some i , then infeasibility is detected. Otherwise, the sequence $\{x_i\}$ produced cannot have strong cluster points.

Note that the condition (5.4a) implies that the feasible sets F_i are such that $F_{i+1} \subset F_i$ (or $\tilde{F}_{i+1} \subset \tilde{F}_i$ for Algorithm 3.1) for all $i \geq 0$. Condition (5.4b) implies that a halfspace generated by projecting x_i onto a set furthest away from x_i is taken. We continue with the proof of Theorem 5.2.

Proof. We consider (1) first. Suppose on the contrary that $\{x_i\}$ has a strong cluster point, say \bar{x} . Assume without loss of generality that K_1 is such that $d(\bar{x}, K_1) = \max_{l \in \{1, \dots, r\}} d(\bar{x}, K_l)$. There must be a point, say x_i , such that $\|x_i - \bar{x}\| < \frac{1}{3}d(\bar{x}, K_1)$. It is elementary to see that $d(x_i, K_1) > 2\|x_i - \bar{x}\|$. So the distance from x_i to a halfspace produced by projecting x_i onto K_1 is at least $2\|x_i - \bar{x}\|$, and this halfspace must not contain \bar{x} . This implies that $\bar{x} \notin \tilde{F}_i$, and that $\bar{x} \notin \tilde{F}_j$ for all $j \geq i$. Therefore \bar{x} cannot be a strong cluster point.

The proof of (3) is the exactly the same as that for (1).

The proof of (2) is similar with slightly different constants. Suppose on the contrary that $\{x_i\}$ has a strong cluster point \bar{x} , where $f(\bar{x}) > 0$. Note that the distance from x_i to the halfspace

$$\{x : f(x_i) + \langle x - x_i, s_i \rangle \leq 0\}$$

is equal to $\frac{1}{\|s_i\|} f(x_i)$. In view of the outer semicontinuity of the subdifferential mapping for convex functions, there is a neighborhood U of \bar{x} such that $\|s\| < \max_{y \in \partial f(\bar{x})} \|y\| + 1$ for all $x \in U$ and $s \in \partial f(x)$. Also, continuity of $f(\cdot)$ implies that $f(x_i)$ can be arbitrarily close to $f(\bar{x})$. If x_i is close enough to \bar{x} such that

$$\|x_i - \bar{x}\| < \frac{1}{2\|s_i\|} f(x_i),$$

then we will get a similar contradiction as before. \square

We improve the techniques in [Pan14b, Theorem 6.2] to prove the following result.

Theorem 5.3. (*Recession directions*) Let $x_0 \in \mathbb{R}^n$, and let $K_l \subset \mathbb{R}^n$ be closed convex sets such that $K := \bigcap_{l=1}^r K_l = \emptyset$. Suppose either

- Algorithm 3.1 is run with $\bar{p} = \infty$ in step 0 and the conditions (5.4) hold, or
- Algorithm 5.1 is run so that the conditions (5.4) hold.

Let $\{x_i\}$ be the sequence produced. Then any cluster point of $\{\frac{x_i}{\|x_i\|}\}_i$ lies in $R(K_l)$, the recession cone of K_l , for all $l \in \{1, \dots, r\}$.

Proof. The proof here is improved from that of [Pan14b, Theorem 6.2]. Let $\{\frac{\tilde{x}_i}{\|\tilde{x}_i\|}\}$ be a subsequence of $\{\frac{x_i}{\|x_i\|}\}$ such that

$$\lim_{i \rightarrow \infty} \|\tilde{x}_i\| = \infty \text{ and } \lim_{i \rightarrow \infty} \frac{\tilde{x}_i}{\|\tilde{x}_i\|} = v. \quad (5.5)$$

We show that such a limit has to lie in $R(K_l)$ for all $l \in \{1, \dots, r\}$. Seeking a contradiction, suppose that v is such that $v \notin R(K_l)$ for some $l \in \{1, \dots, r\}$. Let $L_1 \subset \{1, \dots, r\}$ be such that $v \in R(K_l)$ for all $l \in L_1$, and $L_2 = \{1, \dots, r\} \setminus L_1$.

Claim 1: For each $l \in L_2$, there is a unit vector $w_l \in \mathbb{R}^n$ and $M_l \in \mathbb{R}$ such that $\langle w_l, v \rangle > 0$, and $\langle w_l, c \rangle \leq M_l$ for all $c \in K_l$.

Take any point $y_l \in K_l$. Since $v \notin R(K_l)$, there is some $\gamma \geq 0$ such that $y_l + \gamma v \in K_l$, but $y_l + \gamma' v \notin K_l$ for all $\gamma' > \gamma$. It follows that there exists a unit vector $w_l \in N_{K_l}(y_l + \gamma v)$ such that $\langle w_l, v \rangle > 0$, and we can take $M_l = \langle w_l, y_l + \gamma v \rangle$. This ends the proof of Claim 1.

Claim 2: There is an I such that if $i > I$, $\arg \max_{l \in \{1, \dots, r\}} d(\tilde{x}_i, K_l) \in L_2$.

In view of (5.5), we can write $\tilde{x}_i = \rho_i[v + \beta_i] + \tilde{x}_0$, where $\rho_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}^n$ are such that $\rho_i \rightarrow \infty$ and $\beta_i \rightarrow 0$. For each $l \in L_1$, we can choose $y_l \in K_l$ and have $y_l + \mathbb{R}_+\{v\} \subset K_l$. We can then choose I large enough so that $P_{y_l + \mathbb{R}_+\{v\}}(\tilde{x}_i) = P_{y_l + \mathbb{R}(v)}(\tilde{x}_i)$ for all $i > I$ and $l \in L_1$.

We now estimate $d(\tilde{x}_i, K_l)$ for $l \in L_1$ and i large enough.

$$\begin{aligned} d(\tilde{x}_i, K_l) &\leq d(\tilde{x}_i, y_l + \mathbb{R}(v)) \\ &= d(\rho_i[v + \beta_i] + \tilde{x}_0, y_l + \mathbb{R}(v)) \\ &= d(\rho_i\beta_i + \tilde{x}_0 - y_l, \mathbb{R}(v)) \\ &\leq d(\rho_i\beta_i + \tilde{x}_0 - y_l, 0) \\ &\leq \rho_i\|\beta_i\| + \|\tilde{x}_0 - y_l\|. \end{aligned} \quad (5.6)$$

For $l \in L_2$, let the halfspace H_l be $\{x : \langle w_l, x \rangle \leq M_l\}$. Claim 1 says that we have $K_l \subset H_l$ for some $w_l \in \mathbb{R}^n$ with $\|w_l\| = 1$ such that $\langle w_l, v \rangle > 0$. For i large enough, we have

$$\begin{aligned} d(\tilde{x}_i, K_l) &\geq d(\tilde{x}_i, H_l) \\ &= d(\rho_i[v + \beta_i] + \tilde{x}_0, H_l) \\ &= \langle w_l, \rho_i[v + \beta_i] + \tilde{x}_0 \rangle - M_l \\ &= \rho_i \langle w_l, v \rangle + \rho_i \langle w_l, \beta_i \rangle + \langle w_l, \tilde{x}_0 \rangle - M_l \\ &\geq \rho_i \langle w_l, v \rangle - \rho_i \|\beta_i\| + \langle w_l, \tilde{x}_0 \rangle - M_l. \end{aligned} \quad (5.7)$$

From (5.6) and (5.7), we can estimate that $\arg \max_{l \in \{1, \dots, r\}} d(\tilde{x}_i, K_l) \in L_2$ for all i large enough as needed. This ends the proof of Claim 2.

Let $c_i := P_{K_{l_i^*}}(\tilde{x}_i)$, and let u_i be the unit vector in the direction of $\tilde{x}_i - c_i$. We write $\tilde{x}_i - c_i = \alpha_i u_i$. We have

$$\langle u_i, c_i \rangle = \langle u_i, \tilde{x}_i - \alpha_i u_i \rangle = \langle u_i, \tilde{x}_i \rangle - \alpha_i. \quad (5.8)$$

If i is large enough, there is a tail of the sequence $\{l_i^*\}$ which lies in L_2 . Let l^* be an index in L_2 that appears in the sequence $\{l_i^*\}$ infinitely often. We let $w := w_{l^*}$ and $M = M_{l^*}$. So

$$\alpha_i \langle w, u_i \rangle = \langle w, \tilde{x}_i \rangle - \langle w, c_i \rangle \geq \langle w, \tilde{x}_i \rangle - M.$$

When i is large enough, we have $\alpha_i \langle w, u_i \rangle = \langle w, \tilde{x}_i - c_i \rangle > M - M = 0$, so $\langle w, u_i \rangle > 0$. Hence $\alpha_i \geq \alpha_i \langle w, u_i \rangle \geq \langle w, \tilde{x}_i \rangle - M$. Therefore, making use of (5.8), we have

$$\langle u_i, c_i \rangle \leq \langle u_i, \tilde{x}_i \rangle - \langle w, \tilde{x}_i \rangle + M.$$

By the workings of the corresponding algorithms, we have $\langle u_i, \tilde{x}_i \rangle > \langle u_i, c_i \rangle$ and $\langle u_i, \tilde{x}_j \rangle \leq \langle u_i, c_i \rangle$ for all $j > i$. This gives $\langle u_i, \tilde{x}_j - \tilde{x}_i \rangle \leq 0$, which gives $\langle u_i, v \rangle \leq 0$.

Let u be a cluster point of $\{u_i\}$. We can consider subsequences so that $l_i^* = l^*$ and $\lim_{i \rightarrow \infty} u_i$ exists. For any point $c \in K_{l^*}$, we have

$$\begin{aligned} \langle u, c \rangle &= \lim_{i \rightarrow \infty} \langle u_i, c \rangle \\ &\leq \liminf_{i \rightarrow \infty} \langle u_i, c_i \rangle \\ &\leq \liminf_{i \rightarrow \infty} [\langle u_i, \tilde{x}_i \rangle - \langle w, \tilde{x}_i \rangle + M] \\ &= \liminf_{i \rightarrow \infty} \|\tilde{x}_i\| \left(\left\langle u_i, \frac{\tilde{x}_i}{\|\tilde{x}_i\|} \right\rangle - \left\langle w, \frac{\tilde{x}_i}{\|\tilde{x}_i\|} \right\rangle \right) + M \\ &= \liminf_{i \rightarrow \infty} \|\tilde{x}_i\| [\langle u_i, v \rangle - \langle w, v \rangle] + M \\ &= -\infty, \end{aligned}$$

which is absurd. The contradiction gives $v \in R(K_l)$ for all $l \in \{1, \dots, r\}$. \square

By combining Theorems 5.2 and 5.3, we can conclude the following.

Corollary 5.4. (Certifying infeasibility in finitely many operations) Suppose $X = \mathbb{R}^n$, $K = \emptyset$ and $\cap_{l=1}^r R(K_l) = \emptyset$ in either Algorithm 3.1 run with $\bar{p} = \infty$ in step 0 or Algorithm 5.1. Suppose also that the condition (5.4) is satisfied. For any starting point x_0 , the algorithms terminate with $\tilde{F}_i = \emptyset$ or $F_i = \emptyset$ after finitely many iterations.

Examples involving

$$\begin{aligned} K_1 &= \{(x, y) \in \mathbb{R}^2 \mid y \geq e^{-x}\}, \\ \text{and } K_2 &= \{(x, y) \in \mathbb{R}^2 \mid y \leq -e^{-x}\} \end{aligned}$$

show that when $X = \mathbb{R}^n$, $K = \emptyset$ and $\cap_{l=1}^r R(K_l) \neq \emptyset$, one may not get a certificate of infeasibility in finitely many operations.

Remark 5.5. (Infeasibility certificate) Note that when $X = \mathbb{R}^n$, the normals of the halfspaces need to be linearly dependent before infeasibility can be detected. Since a set of n vectors in \mathbb{R}^n can be arbitrarily close to another set of n linearly independent vectors, one might need a large number of halfspaces to detect infeasibility. The QP algorithm in [GI83] for example gives a (Farkas Lemma type) certificate of infeasibility of a system $Ax \leq b$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, by finding a vector $r \in \mathbb{R}^n$ such that

$$r \in \mathbb{R}_+^n, r^T A = 0 \text{ and } r^T b < 0. \quad (5.9)$$

An acceptable relaxation of $r^T A = 0$ would be that $r^T A$ being approximately zero.

We remark on aggregation of constraints.

Remark 5.6. (Aggregation of constraints) Observe that, in Algorithm 5.1, x_i is the projection of x_0 onto F_{i-1} . We can aggregate some of the constraints describing F_{i-1} in a manner similar to bundle methods. More precisely, for constraints $a_j^T x \leq b_j$ for all $j \in J$, we can find multipliers $\lambda_j \geq 0$ for all $j \in J$ such that we now store the single constraint

$$\left[\sum_{j \in J} \lambda_j a_j \right]^T x \leq \left[\sum_{j \in J} \lambda_j b_j \right].$$

The polyhedron thus produced would be larger than F_{i-1} , but fewer linear constraints need to be stored. We only require that x_i is the projection of x_0 onto the new polyhedron. Projecting x_i onto a set K_l not containing x_i produces a new polyhedron, from which we get $\|x_{i+1} - x_0\| > \|x_i - x_0\|$.

We present another result for detecting infeasibility in the BAP.

Theorem 5.7. (BAP) *Let X be a Hilbert space, $x_0 \in X$ and $K_l \subset X$ be closed convex sets for $l \in \{1, \dots, r\}$. Suppose Algorithm 5.1 is modified such that*

- (1) *The aggregation of constraints mentioned in Remark 5.6 is carried out, and that x_i is still the projection of x_0 onto the new polyhedron.*
- (2) *Halfspaces obtained by projecting x_i onto some of the K_l are intersected with the polyhedron stored earlier. One of these halfspaces is obtained by projecting x_i onto the set $K_{l_i}^*$ for which $d(x_i, K_{l_i}^*) = \max_{l \in \{1, \dots, r\}} d(x_i, K_l)$. The next iterate x_{i+1} is the projection of x_0 onto this new polyhedron.*

Then the sequence of iterates $\{x_i\}$ cannot have a strong cluster point. If $X = \mathbb{R}^n$, the sequence $\{\|x_i - x_0\|\}_i$ must be monotonically increasing to infinity.

Proof. Suppose on the contrary that \bar{x} is a strong cluster point. Without loss of generality, let K_1 be such that $d(\bar{x}, K_1) = \max_{l \in \{1, \dots, r\}} d(\bar{x}, K_l)$. If $\|x_i - x_0\| > \|\bar{x} - x_0\|$, then $\|x_j - x_0\| > \|\bar{x} - x_0\|$ for all $j \geq i$, which means that \bar{x} cannot be a cluster point. Suppose x_i is close enough to \bar{x} so that

$$\|x_i - x_0\|^2 + [d(\bar{x}, K_1) - \|x_i - \bar{x}\|]^2 > \|\bar{x} - x_0\|^2.$$

Then $d(x_i, K_1) \geq d(\bar{x}, K_1) - \|x_i - \bar{x}\|$. In other words, the distance of x_i to the halfspace produced by projecting x_i onto K_1 is at least $d(\bar{x}, K_1) - \|x_i - \bar{x}\|$.

In view of (1), we have $\angle x_{i+1} x_i x_0 \geq \pi/2$. It is elementary to check that

$$\|x_{i+1} - x_0\|^2 \geq \|x_i - x_0\|^2 + [d(\bar{x}, K_1) - \|x_i - \bar{x}\|]^2 > \|\bar{x} - x_0\|^2,$$

which would imply that \bar{x} cannot be a cluster point of $\{x_i\}$. The second sentence of the theorem is easy. \square

6. PRACTICAL IMPLEMENTATION USING A DUAL QP ALGORITHM

The algorithms in this paper have been shown to enjoy several useful theoretical properties mentioned in earlier parts of this paper. However, the size of the QPs that need to be solved may be too large in practice for a practical QP solver. In this section, we explain that the dual active set QP algorithm of Goldfarb and Idrani [Gl83] can give useful iterates even when the QPs are not solved to optimality.

For the sake of simplicity, we repeat the narrative of [Gl83] where only inequality constraints are considered. Let $C \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^m$ and consider the QP that arises repeatedly in our algorithms

$$\begin{aligned} QP(y, C, b) := \min_{\tilde{x} \in \mathbb{R}^n} \quad & \frac{1}{2} \|\tilde{x} - y\|^2 \\ \text{s.t.} \quad & C^T \tilde{x} \leq b. \end{aligned}$$

In other words, we want to project the point y onto the polyhedron $F := \{\tilde{x} : C^T \tilde{x} \leq b\}$. A positive definite Hessian in the QP is required for the algorithm in [Gl83] to work, which is indeed met in our context, where the Hessian of our QP is the identity matrix. At the projection $P_F(y)$, not every constraint in $C^T \tilde{x} \leq b$ is tight.

The dual QP algorithm of [Gl83] starts with a candidate active index set $A_0 := \emptyset$ and $\tilde{x}_0 := y$. Note that $\tilde{x}_i = P_{F_{A_i}}(y)$ for $i = 0$, where

$$F_{A_i} := \{\tilde{x} : c_j^T \tilde{x} \leq b_j \text{ for all } j \in A_i\},$$

with c_j being the j th column of C . The structure $\tilde{x}_i = P_{F_{A_i}}(y)$ and $c_j^T \tilde{x}_i = b_j$ for all $j \in A_i$ would be maintained throughout the algorithm over all iterations i until termination at an optimal active set \bar{A} , where $P_{F_{\bar{A}}}(y) = P_F(y)$. In each iteration i , the next index set A_i is determined by first choosing some $j \notin A_{i-1}$ such that $c_j^T \tilde{x}_i > b_j$. Next, the active set A_i is updated so that $A_i \subset A_{i-1} \cup \{j\}$. Some useful properties are:

- (1) $\|\tilde{x}_i - y\|$ is monotonically increasing, and
- (2) The dual QP algorithm converges in finitely many iterations to $P_F(y)$.

Suppose at the i th iteration of the algorithms in earlier sections, we want to project x_i onto some polyhedron F . We would let $\tilde{x}_0 := x_i$, and run the GI algorithm to get a sequence of iterates $\{\tilde{x}_j\}_j$ that converges to $P_F(\tilde{x}_0) = P_F(x_i)$ in finitely many steps by property (2). This convergence property is reassuring, but the number of iterations may still be prohibitively large. We now show that the iterates $\{\tilde{x}_j\}_j$ get better for the associated feasibility problem, even if we don't arrive at $P_F(\tilde{x}_0)$. Since $\tilde{x}_j = P_{F_{A_j}}(\tilde{x}_0)$ and $F \subset F_{A_j}$, Fejér monotonicity (2.1) implies that

$$\|\tilde{x}_j - c\|^2 \leq \|x_i - c\|^2 - \|x_i - \tilde{x}_j\|^2 \text{ for all } c \in F. \quad (6.1)$$

Property (1) implies that the term $\|x_i - \tilde{x}_j\|^2$ increases with the number of iterations j in the GI algorithm, so (6.1) implies that $\|\tilde{x}_j - c\|^2$ decreases for all $c \in F$, and by at least the factor $\|x_i - \tilde{x}_j\|^2$. In other words, the iterates \tilde{x}_j get better for the associated SIP, CIP or BAP. For the SIP and CIP, we may arrive at a point satisfying the conditions in step 2 of Algorithm 3.1 or 4.1 without solving the QP to optimality by considering $x_i + t(\tilde{x}_j - x_i)$ for some $t \in [1, 2]$. Recall that our earlier results tell us that such a point can still give multiple-term superlinear convergence.

We call the inner iterations to solve the QPs *inner GI steps*. The inner GI steps in the dual QP algorithm allows for the underlying QP in the BAP to be solved using warmstart solutions from previous iterations. In other words, the QPs do not have to be solved from scratch.

Note that for the BAP, we project from x_0 all the time, but in the SIP, we project from x_i at the i th iteration, and x_i is a point closer to $K := \cap_{l=1}^r K_l$ than x_0 : We think this is a reason why the SIP is easier to solve than the BAP.

In prevailing SIP algorithms, the operations that can be taken are (1) to find supporting halfspaces of K_l by projecting from x_i , and (2) to move x_i to a point x_{i+1} by various strategies. We propose a new operation: (3) to perform inner GI steps to find better candidates for x_{i+1} before performing operation (2). By introducing operation (3), we can reduce the SIP for sets with smooth boundaries to Newton-like methods that give superlinear convergence in the manner of [GP98, GP01] or of the algorithms in this paper, and such fast convergence had indeed been observed. Further details are discussed in [Pan13].

7. CONCLUSION

We have done what we set out to do in Subsection 1.1. The SIP and the CIP can be cast as a problem of finding an x such that

$$\max_{l \in \{1, \dots, r\}} f_l(\cdot) \leq 0$$

or to give a certificate of nonexistence if no such x exists, where each $f_l(\cdot)$ is convex. Our algorithms here can achieve multiple-term superlinear or multiple-term quadratic convergence if the proper conditions hold. The BAP (5.1) cannot be written in this form, which may be why we cannot expect the fast convergence for the BAP in general. We also make further observations on the infeasible case in Section 5, showing that under reasonable conditions, a finite number of operations can give a certificate of infeasibility for both the SIP and BAP.

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Current address: Department of Mathematics, National University of Singapore, Block S17 08-11,
10 Lower Kent Ridge Road, Singapore 119076

E-mail address: matpchj@nus.edu.sg